SymPy: Symbolic Mathematics in Pure Python

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March 2, 2011

Presentation plan

- Introduction to SymPy
  - What is SymPy and why we need it?
  - Pure Python pros and cons
  - List of features
  - How to contribute
- Examples
  - Graph k–coloring with Gröbner bases

What is SymPy?

- A pure Python library for symbolic mathematics

```
>>> from sympy import *
>>> x = Symbol('x')

>>> limit(sin(pi*x)/x, x, 0)
pi

>>> integrate(x + sinh(x), x)
(1/2)*x**2 + cosh(x)

>>> diff(_, x)
x + sinh(x)
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Symbolic capabilities (1)

Lets consider the following function (Gruntz, 1996):

$$f = x^{\left(1-\log\left(\log\left(\log\left(\log\left(\frac{1}{x}\right)\right)\right)\right)\right)}$$

We would like to compute the following limit:

$$\lim_{x \to 0^+} f(x) = ?$$

Lets try numerical approach:

k	1	2	3	4	5
$O(f(10^{-10^k}))$	10^{-9}	10^{-48}	10^{-284}	10^{-1641}	10^{-7836}

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We can use SymPy to prove this guess wrong:

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In [1]: from sympy import var, log, limit

In [2]: var('x')
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In [3]: f = x**(1 - log(log(log(log(1/x)))))

In [4]: limit(f, x, 0)
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Why reinvent the wheel for the 37th time?

There are numerous symbolic manipulation systems:

- Proprietary software:
  - Mathematica, Maple, Magma, ...
- Open Source software:
  - AXIOM, GiNaC, Maxima, PARI, Sage, Singular, Yacas, ...

Problems:

- all invent their own language
  - need to learn yet another language
  - separation into core and library
  - hard to extend core functionality
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What does pure Python mean?

- simply download and start computing

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$ git clone git://github.com/sympy/sympy.git
$ cd sympy
$ bin/isympy
```

- no dependencies by default (besides Python)
  - works under Python 2.4, 2.5, 2.6, 2.7
  - support for Python 3.x under development
  - extra dependencies allowed for additional features
    - gmpy, Cython — speed improvement
    - Pyglet, Matplotlib — 2D & 3D plotting
    - IPython — interactive sessions
- preview every algorithm implemented in SymPy
  - `source(obj)`, `obj??`

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Issues with pure Python approach (1)

- you have to define symbols before using them

```
In [1]: t
(...)
NameError: name 't' is not defined

In [2]: var('t')
Out[2]: t

In [3]: symbols('a0:5')
Out[3]: (a0, a1, a2, a3, a4)
```

- 1/3 is not what you may expect

```
In [3]: 1/3
Out[3]: 0.333333333333

In [4]: Rational(1, 3)
Out[4]: 1/3

In [5]: S("1/3")
Out[5]: 1/3
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Issues with pure Python approach (2)

- ^ is not exponentiation operator

```
In [6]: 2^3
Out[6]: 1

In [7]: 2**3
Out[7]: 8
```

- large(er) computations may require tweaking Python

```
In [8]: f = Poly(range(100), x)

In [9]: horner(f)
Out[9]:
(...)
RuntimeError: maximum recursion depth exceeded

In [10]: %time _ = horner(f)
CPU times: user 0.01 s, sys: 0.00 s, total: 0.01 s
Wall time: 0.01 s
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List of SymPy's modules (1)

assumptions	assumptions engine
concrete	symbolic products and summations
core	basic class structure: Basic, Add, Mul, Pow, ...
functions	elementary and special functions
galgebra	geometric algebra
geometry	geometric entities
integrals	symbolic integrator
interactive	interactive sessions (e.g. IPython)
logic	boolean algebra, theorem proving
matrices	linear algebra, matrices
mpmath	fast arbitrary precision numerical math

List of SymPy's modules (2)

ntheory	number theoretical functions
parsing	Mathematica and Maxima parsers
physics	physical units, quantum stuff
plotting	2D and 3D plots using Pyglet
polys	polynomial algebra, factorization
printing	pretty-printing, code generation
series	symbolic limits and truncated series
simplify	rewrite expressions in other forms
solvers	algebraic, recurrence, differential
statistics	standard probability distributions
utilities	test framework, compatibility stuff

How to get involved?

- Visit our main web site:
  - `www.sympy.org`
- and additional web sites:
  - `docs.sympy.org`
  - `wiki.sympy.org`
  - `live.sympy.org`
- Contact us on our mailing list:
  - `sympy@googlegroups.com`
- or/and IRC channel:
  - `#sympy` on FreeNode
- Clone source repository:

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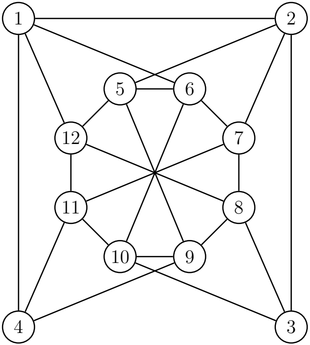
Graph k–coloring with Gröbner bases (1)



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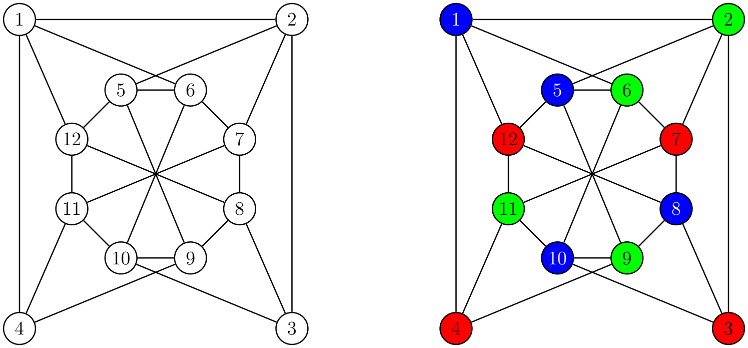

Graph k–coloring with Gröbner bases (2)

Given a graph $\mathcal{G}(V, E)$ we write two sets of equations:

- I_k — allow one of k colors per vertex

$$I_k = \{x_i^k - 1 : i \in V\}$$

- $I_{\mathcal{G}}$ — adjacent vertices have different colors assigned

$$I_{\mathcal{G}} = \{x_i^{k-1} + x_i^{k-2}x_j + \ldots + x_i x_j^{k-2} + x_j^{k-1} : (i, j) \in E\}$$

Next we solve $I_k \cup I_{\mathcal{G}}$ using the Gröbner bases method.

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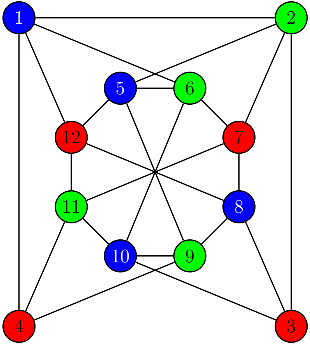
Graph k–coloring with Gröbner bases (3)

$$\{x_1 + x_{11} + x_{12},$$
$$x_2 - x_{11},$$
$$x_3 - x_{12},$$
$$x_4 - x_{12},$$
$$x_5 + x_{11} + x_{12},$$
$$x_6 - x_{11},$$
$$x_7 - x_{12},$$
$$x_8 + x_{11} + x_{12},$$
$$x_9 - x_{11},$$
$$x_{10} + x_{11} + x_{12},$$
$${x_{11}}^2 + x_{11}x_{12} + x_{12}^2,$$
$${x_{12}}^3 - 1\}$$



Graph k–coloring with Gröbner bases (4)

Here is how to solve 3–coloring problem in SymPy:

```
In [1]: V = range(1, 12+1)
In [2]: E = [(1,2),(2,3),(1,4),(1,6),(1,12),(2,5),(2,7),
(3,8),(3,10),(4,11),(4,9),(5,6),(6,7),(7,8),(8,9),(9,10),
(10,11),(11,12),(5,12),(5,9),(6,10),(7,11),(8,12)]

In [3]: X = [ Symbol('x' + str(i)) for i in V ]
In [4]: E = [ (X[i-1], X[j-1]) for i, j in E ]

In [5]: I3 = [ x**3 - 1 for x in X ]
In [6]: Ig = [ x**2 + x*y + y**2 for x, y in E ]

In [7]: G = groebner(I3 + Ig, X, order='lex')

In [8]: G != [1]
Out[8]: True
```

Graph k–coloring with Gröbner bases (4)

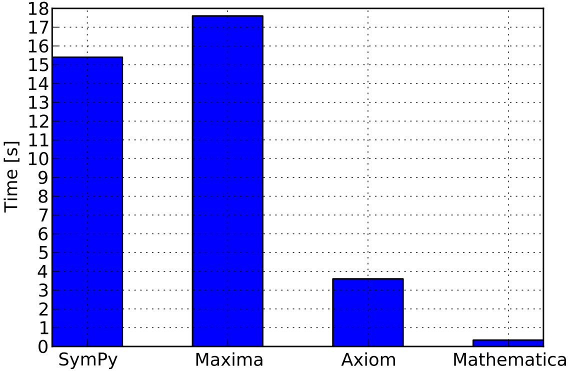


Figure: Average timing of Gröbner basis computation

Thank you for your attention!

Questions, remarks, discussion ...