A semi-classical approach for Lyapunov exponents of a quantum mechanical system

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Boulder, 1998

Problem: how to define a Lyapunov exponent for a quantum system?

Idea: use a particle-like approach for the quantum system (Bohm, 1952).

Consider a 2-d system with ground state wavefunction $\psi(x,y)$, and write

$$\psi(x,y) = N \exp(-W(x,y)/\hbar)$$

Using Schrödinger's equation

$$[-\frac{\hbar}{2m}\nabla^2 + V(x,y)]\psi(x,y) = E\psi(x,y)$$

one can then obtain

$$U(x,y) \equiv V - E = \frac{1}{2}\hbar^2 \frac{\nabla^2\psi(x,y)}{\psi(x,y)}$$

It is customary to define

$$\Phi(x,y) = \frac{1}{\sqrt{2}}\nabla W(x,y)$$

which gives U in the form

$$U(x,y) = \Phi^2(x,y) - \frac{\hbar}{\sqrt{2}}\nabla \cdot \Phi(x,y)$$

or equivalently

$$U(x,y) = U_{tree}(x,y) + U_{loop}(x,y)$$

Here, we'll then keep only U_{tree} and attempt to construct a classical analysis of our system. However, since U_{tree} is a quantum object

$$U_{tree}(x,y) = \frac{\hbar^2}{2}\left[\frac{\nabla\psi(x,y)}{\psi(x,y)}\right]^2$$

this description will be different from a purely classical one.

Another way to motivate this: Supersymmetry inspired arguments show that using U_{tree} one can obtain a highly improved WKB-type approximation.

Program: first solve the quantum problem (ground state), then the classical one with U_{tree}, which "knows" about quantum behavior.

Quantum solution: Finite Element solver

- Variational type method, but with a *local* basis instead of a global one used in normal variational calculations.

- Very accurate, but fairly expensive computationally.

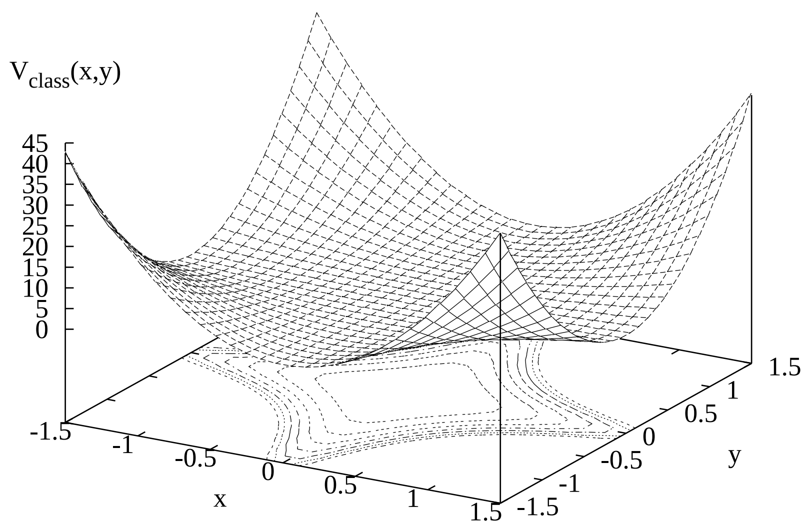
- Provides a continuous and differentiable solution: important for this problem because $U_{tree}(x,y)$ needs $\nabla\psi(x,y)$.

Classical problem: a fifth order adaptive Runge-Kutta (Numerical Recipes), with finite element routines for the calculation of U_{tree}.

System: a 2-d isotropic oscillator plus a non-linear coupling

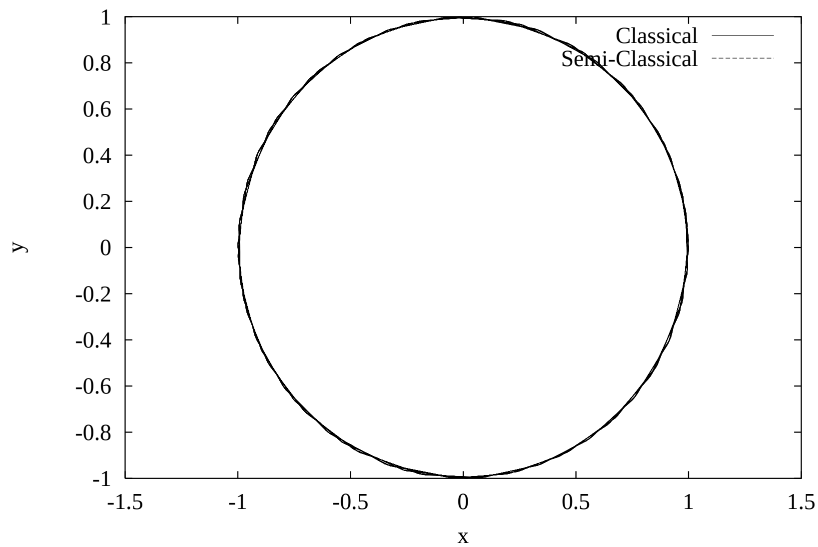
$$V(x,y) = \frac{1}{2}(x^2 + y^2) + 4kx^2y^2$$

A typical plot of $V(x,y)$, for $k = 2$

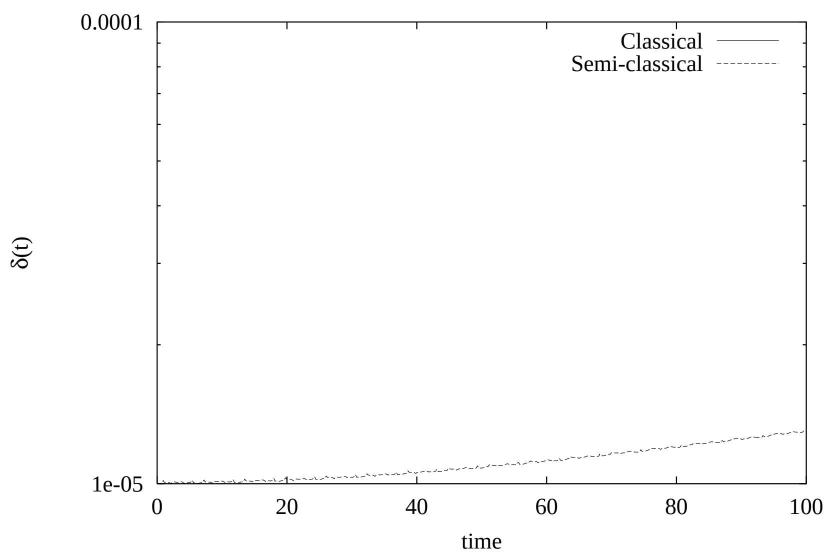


The system is obviously regular (soluble) for $k = 0$ but has no known analytical solution for $k \neq 0$.

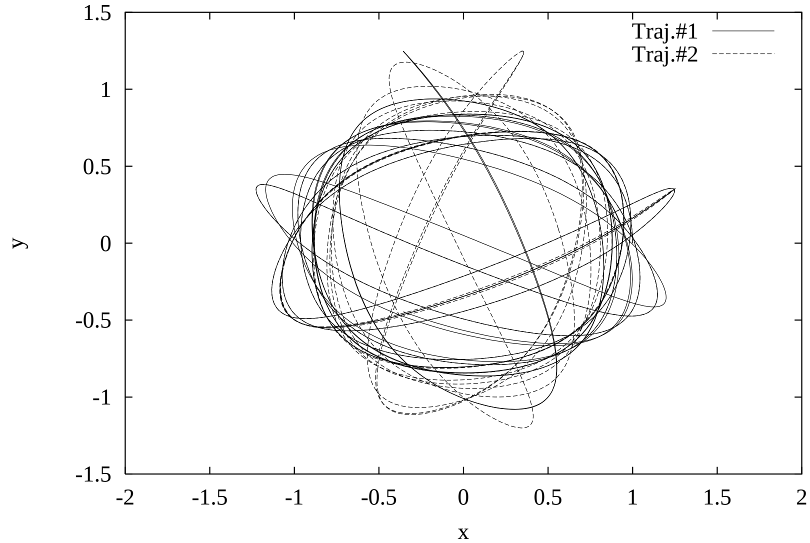
The trajectories in the linear ($k = 0$) case



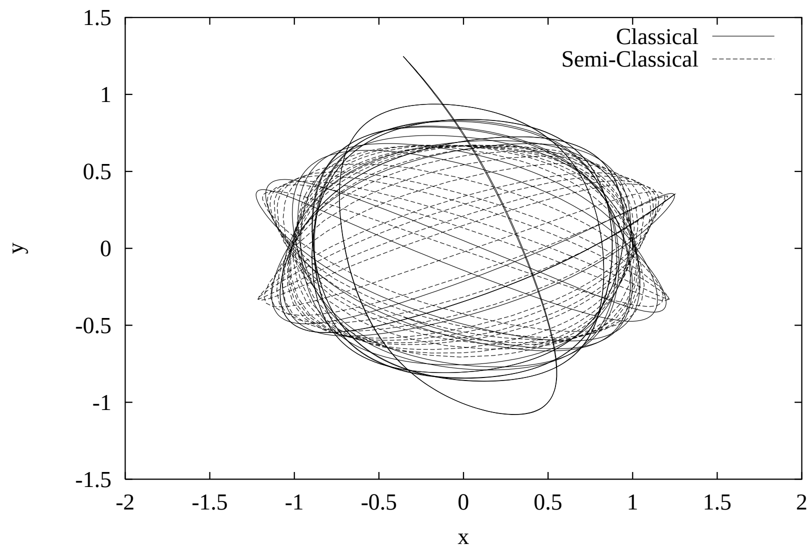
And the *phase space* separation $\delta(t)$



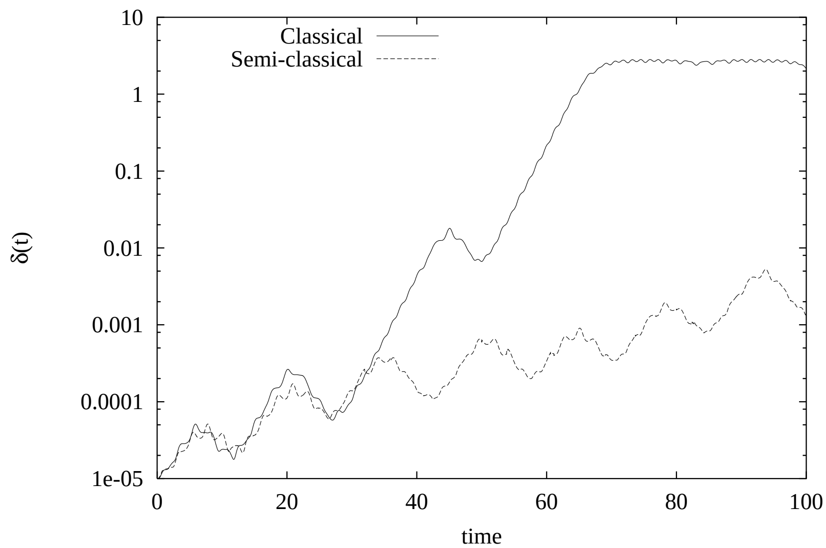
For $k = 0.2$, two nearby classical orbits



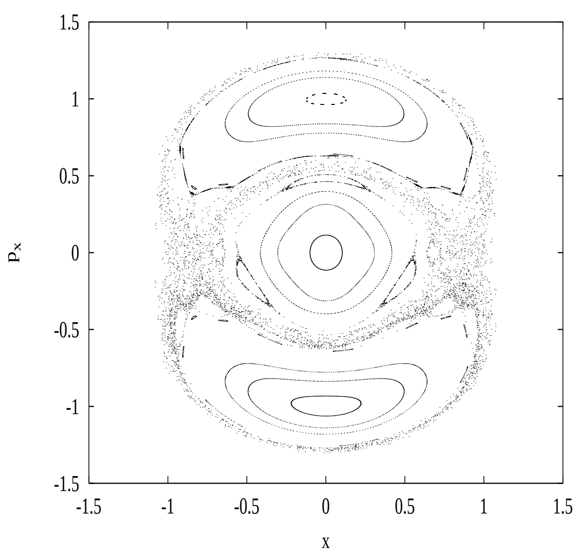
and classical vs. semi-classical orbits



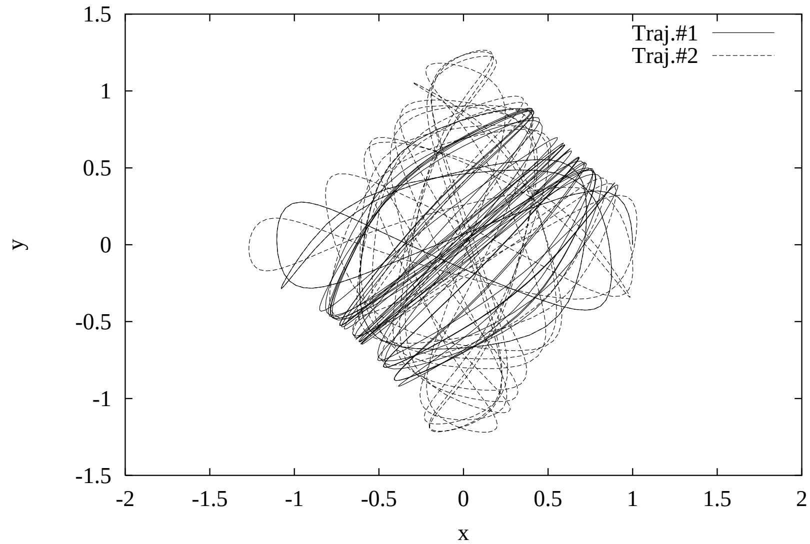
The divergence of the trajectories



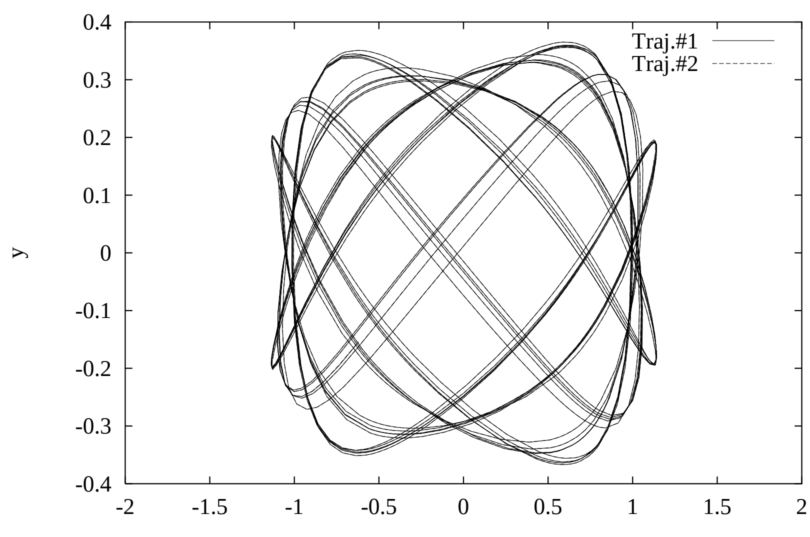
and the classical Poincaré section



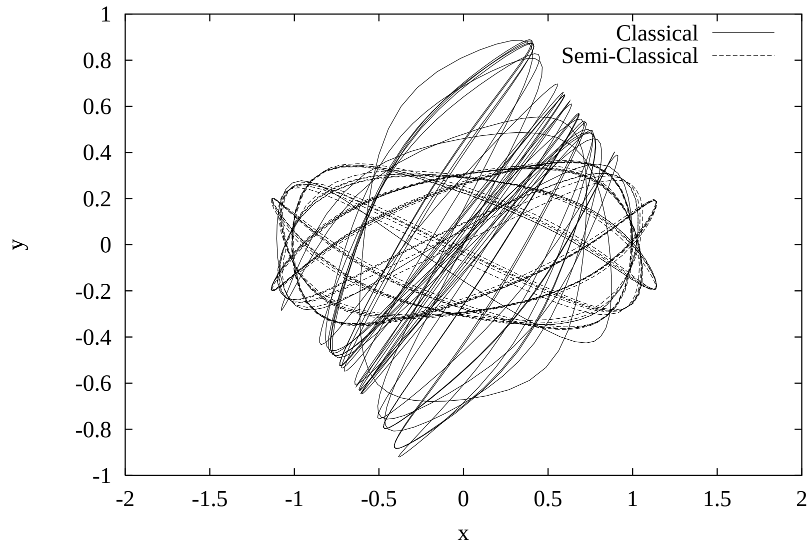
These effects are stronger for $k = 1$. Classical:



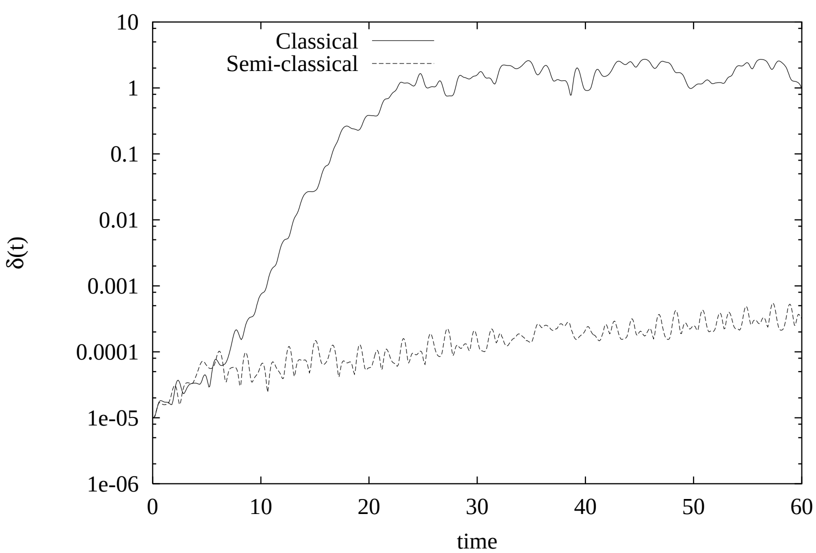
And "quantum" (semi-classical)



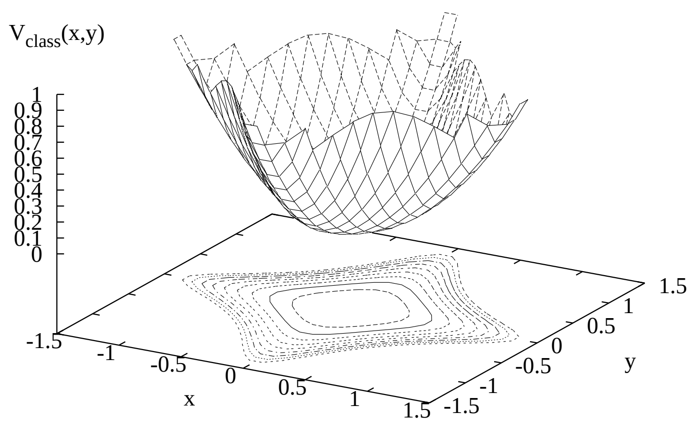
The two types of orbit evolve differently

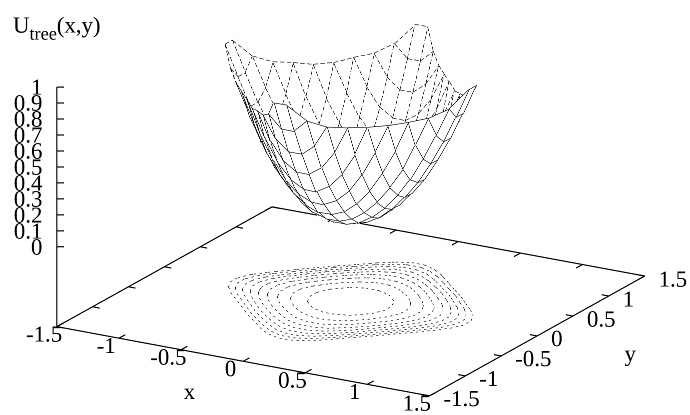


The phase space distance $\delta(t)$ shows it well

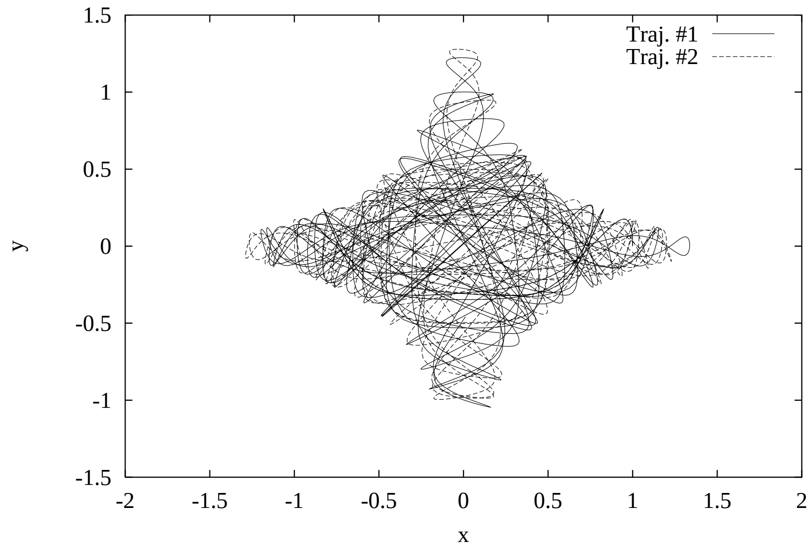


This behavior comes from the marked difference between $V_{class}(x,y)$ and $U_{tree}(x,y)$

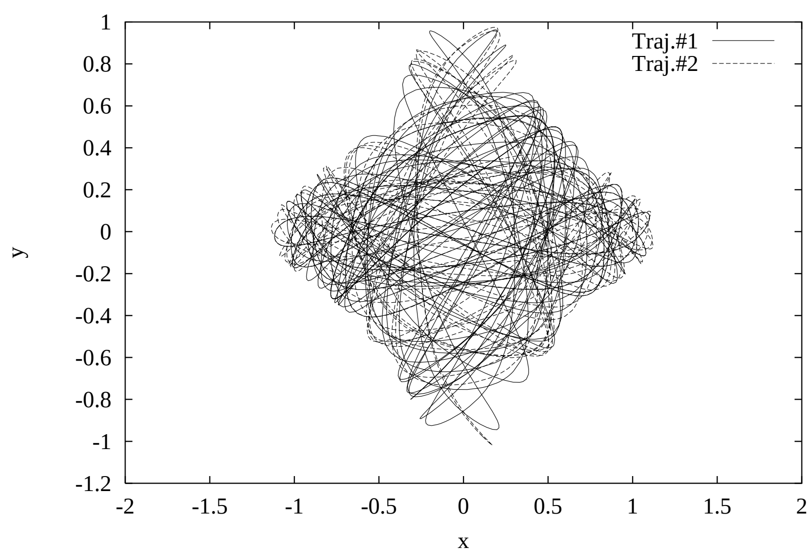




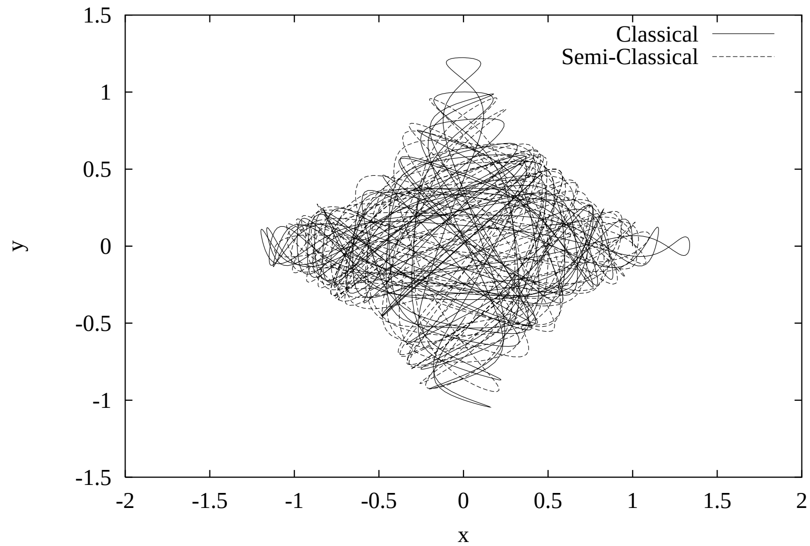
At $k = 4$ both systems are chaotic. Classical:



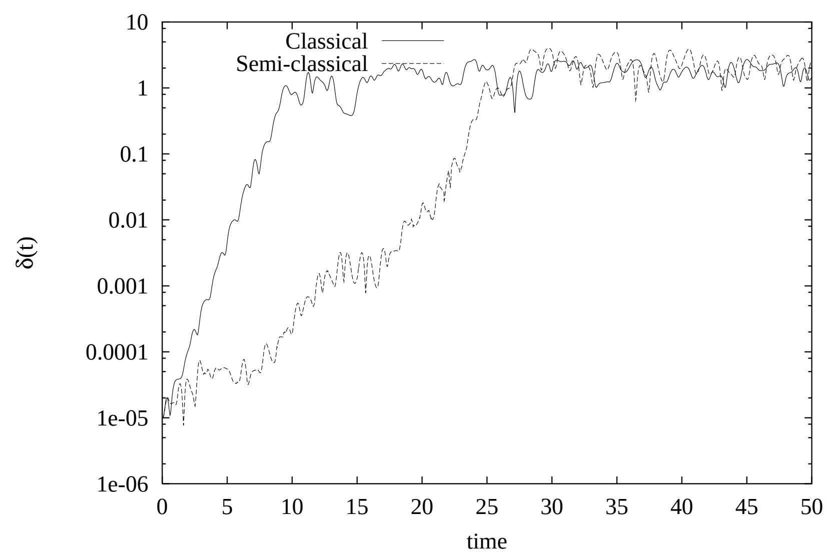
and quantum orbits:



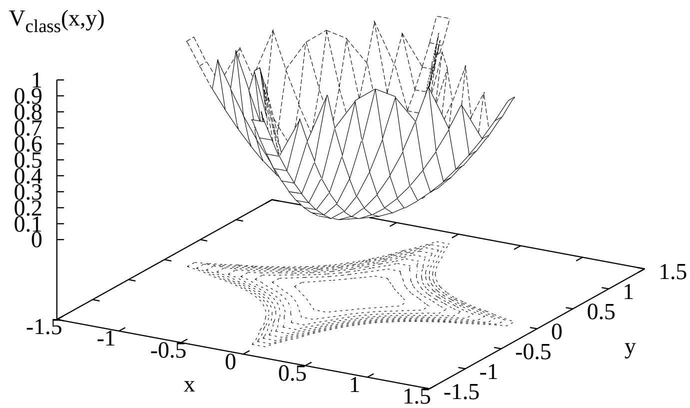
A side by side comparison is instructive

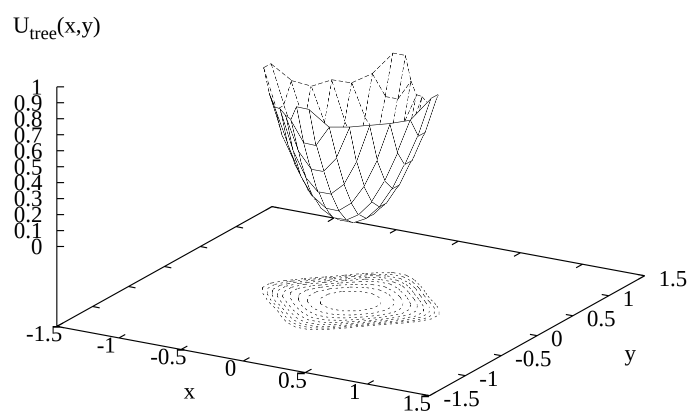


Still some quantum suppression of chaos

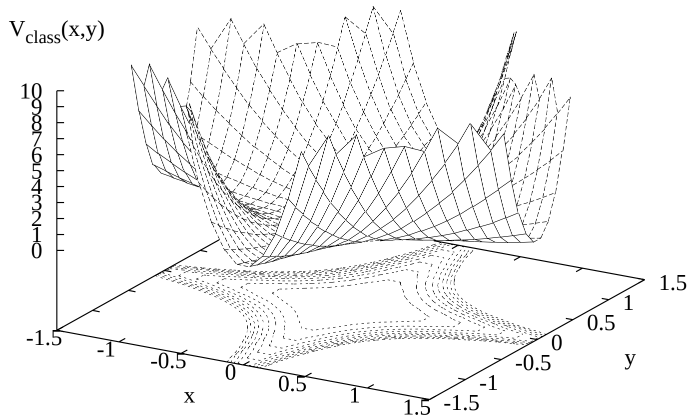


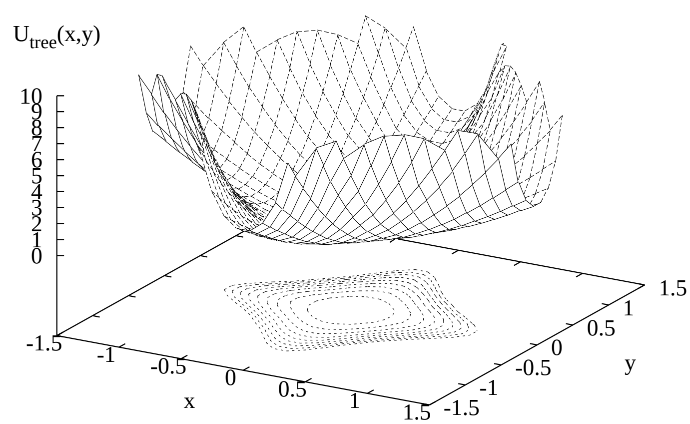
The difference in the potentials is quite clear



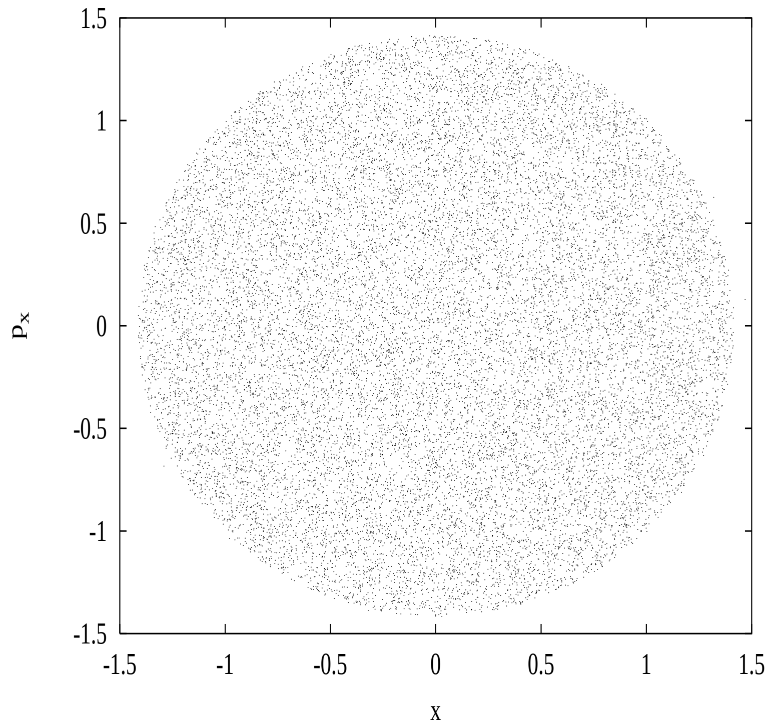


And with a different vertical scale

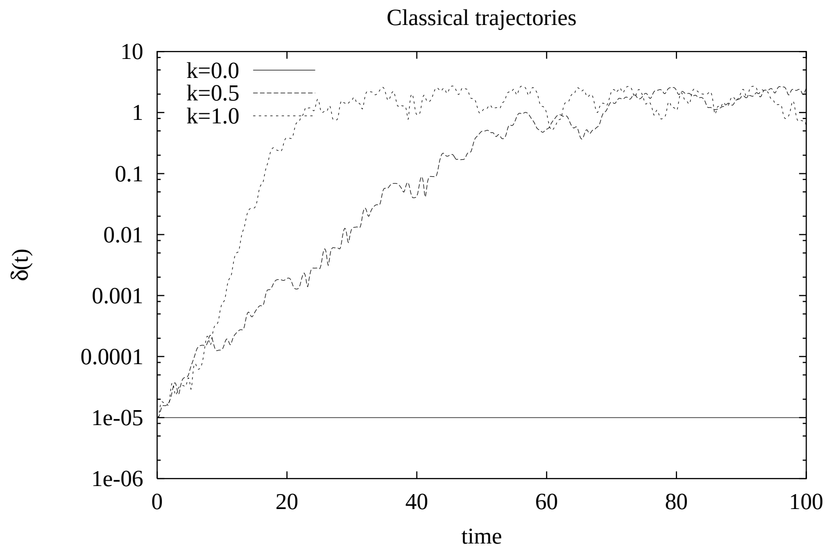




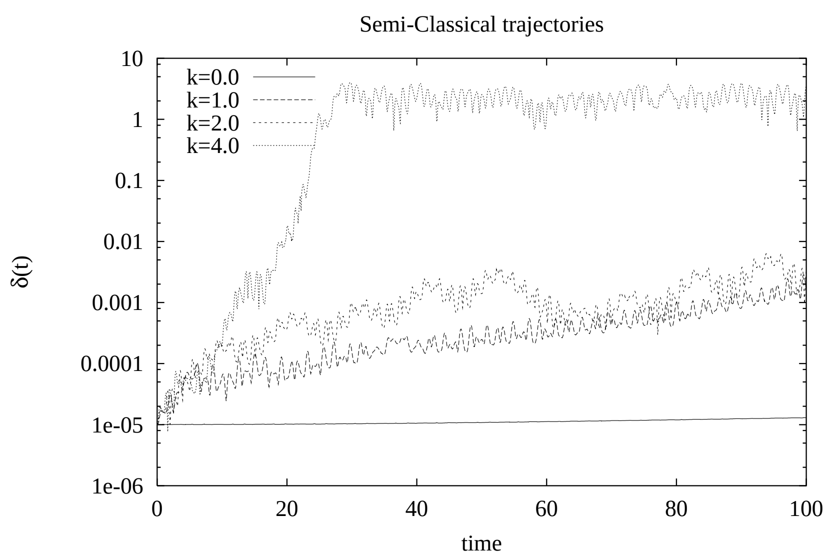
At $k = 4$the classical system is fully chaotic, as the Poincaré surface of section shows well



A global look at orbit divergence. Classical



and semi-classical



Conclusions and outlook

- U_{tree} provides non-trivial quantum corrections to the classical behavior.

- A quantum suppression of chaos is observed, but not a complete elimination.

- A Poincaré section study of U_{tree} would be useful to clarify how phase space is globally affected by the quantum effects.

- A 2-d WKB? Interesting but immensely difficult in fully chaotic regimes: one needs *all* periodic orbits of the system (they are dense in phase space).

- However, *maybe* for small k the quantum suppression of chaos is enough to allow the WKB approach. Interesting...