

A semi-classical approach for Lyapunov exponents of a quantum mechanical system

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Problem: how to define a Lyapunov exponent for a quantum system?

Idea: use a particle-like approach for the quantum system (Bohm, 1952).

Consider a 2- d system with ground state wavefunction $\psi(x, y)$, and write

$$\psi(x, y) = N \exp(-W(x, y)/\hbar)$$

Using Schrödinger's equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x, y)\right]\psi(x, y) = E\psi(x, y)$$

one can then obtain

$$U(x, y) \equiv V - E = \frac{1}{2}\hbar^2 \frac{\nabla^2\psi(x, y)}{\psi(x, y)}$$

It is customary to define

$$\Phi(x, y) = \frac{1}{\sqrt{2}}\nabla W(x, y)$$

which gives U in the form

$$U(x, y) = \Phi^2(x, y) - \frac{\hbar}{\sqrt{2}} \nabla \cdot \Phi(x, y)$$

or equivalently

$$U(x, y) = U_{tree}(x, y) + U_{loop}(x, y)$$

Here, we'll then keep only U_{tree} and attempt to construct a classical analysis of our system. However, since U_{tree} is a quantum object

$$U_{tree}(x, y) = \frac{\hbar^2}{2} \left[\frac{\nabla \psi(x, y)}{\psi(x, y)} \right]^2$$

this description will be different from a purely classical one.

Another way to motivate this: Supersymmetry inspired arguments show that using U_{tree} one can obtain a highly improved WKB-type approximation.

Program: first solve the quantum problem (ground state), then the classical one with U_{tree} , which “knows” about quantum behavior.

Quantum solution: Finite Element solver

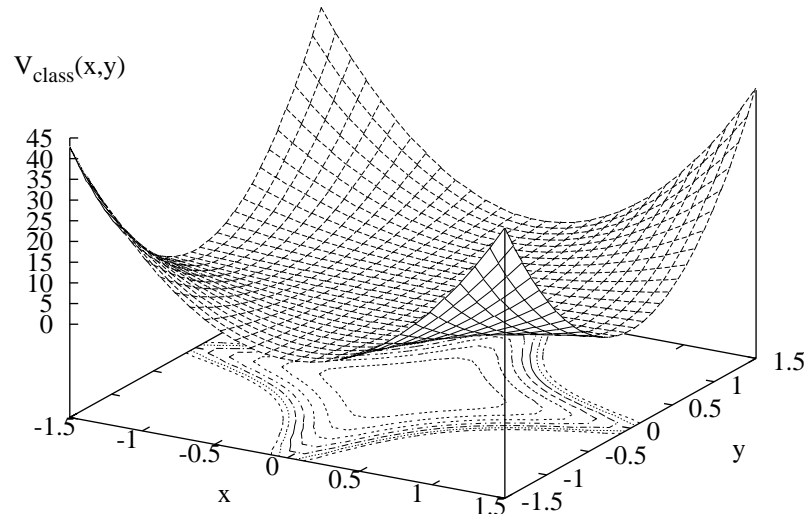
- Variational type method, but with a *local* basis instead of a global one used in normal variational calculations.
- Very accurate, but fairly expensive computationally.
- Provides a continuous and differentiable solution: important for this problem because $U_{tree}(x, y)$ needs $\nabla\psi(x, y)$.

Classical problem: a fifth order adaptive Runge-Kutta (Numerical Recipes), with finite element routines for the calculation of U_{tree} .

System: a 2- d isotropic oscillator plus a non-linear coupling

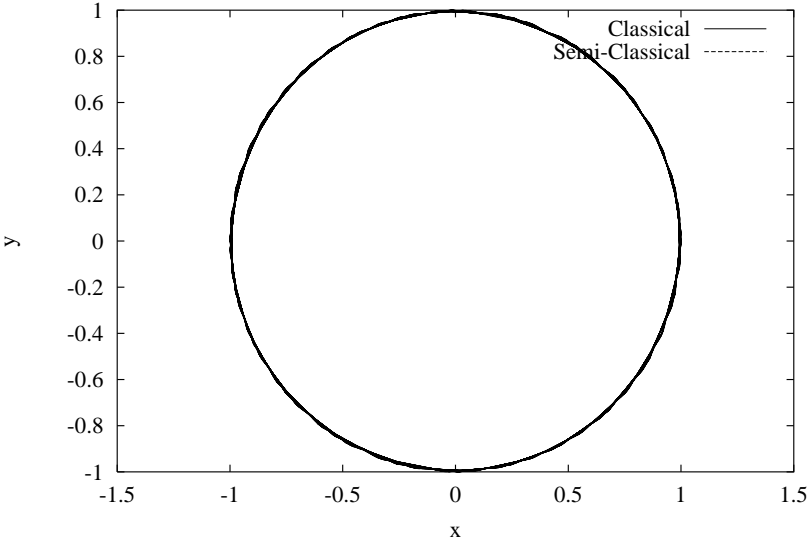
$$V(x, y) = \frac{1}{2}(x^2 + y^2) + 4kx^2y^2$$

A typical plot of $V(x, y)$, for $k = 2$

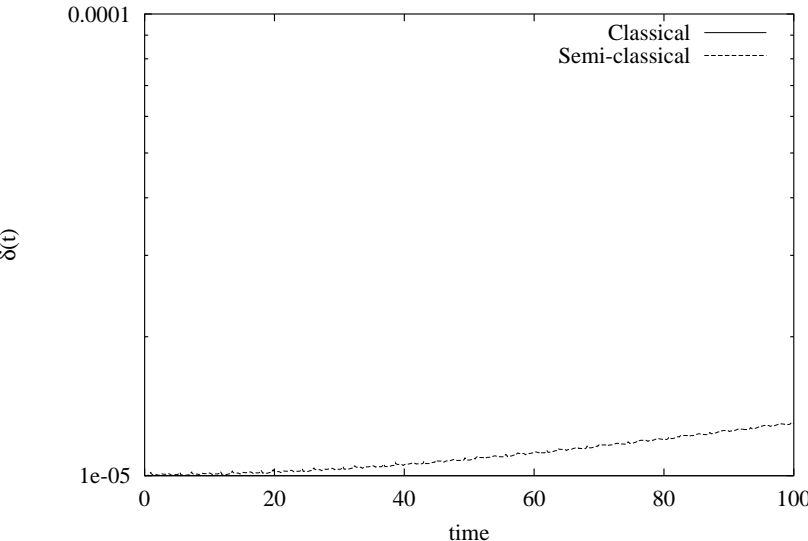


The system is obviously regular (soluble) for $k = 0$ but has no known analytical solution for $k \neq 0$.

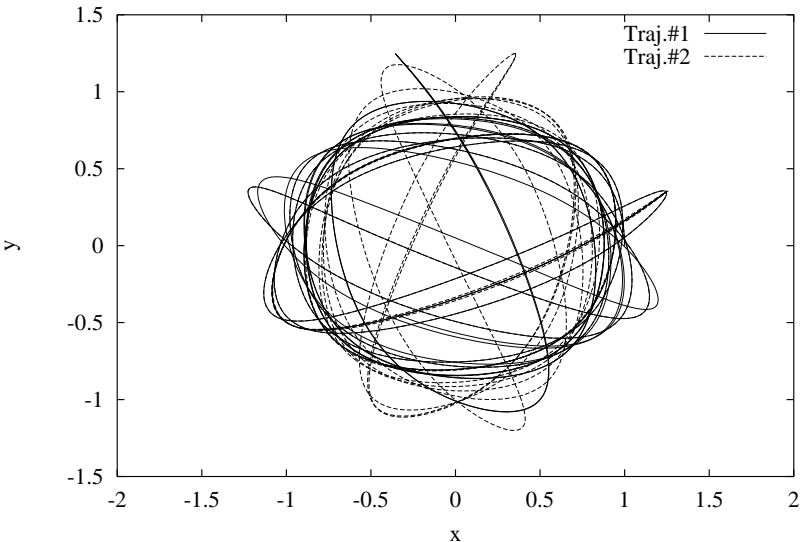
The trajectories in the linear ($k = 0$) case



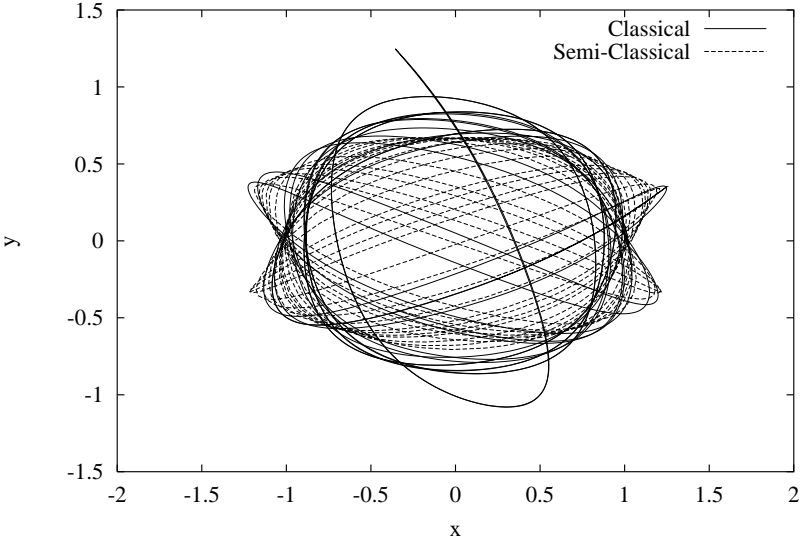
And the *phase space separation* $\delta(t)$



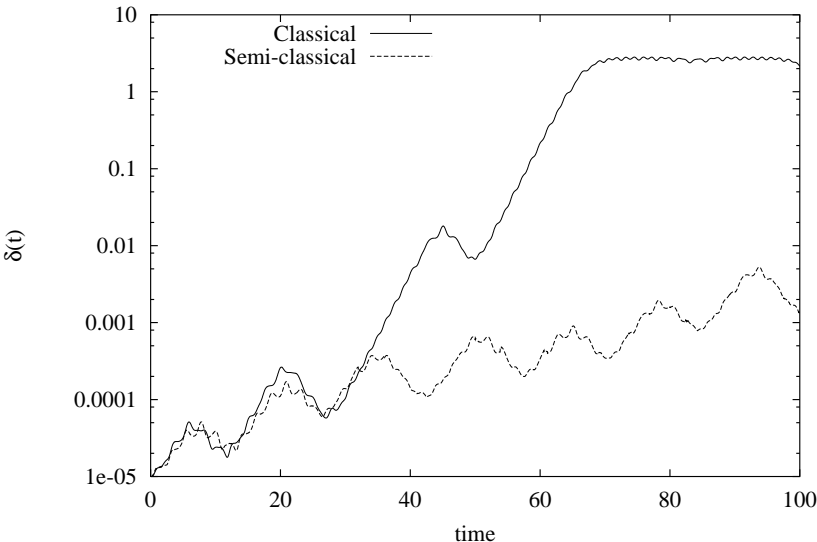
For $k = 0.2$, two nearby classical orbits



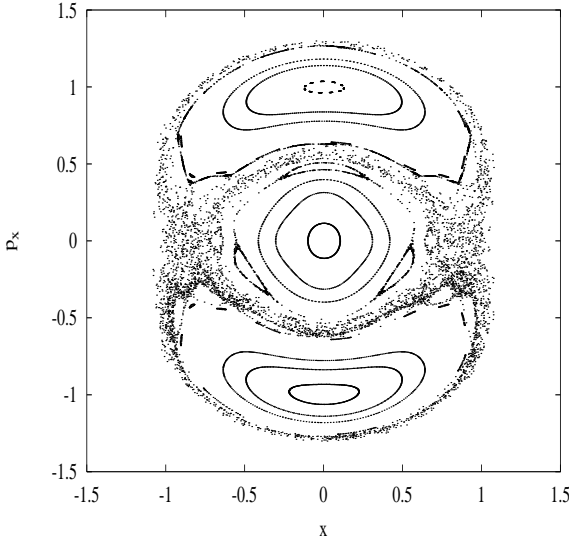
and classical vs. semi-classical orbits



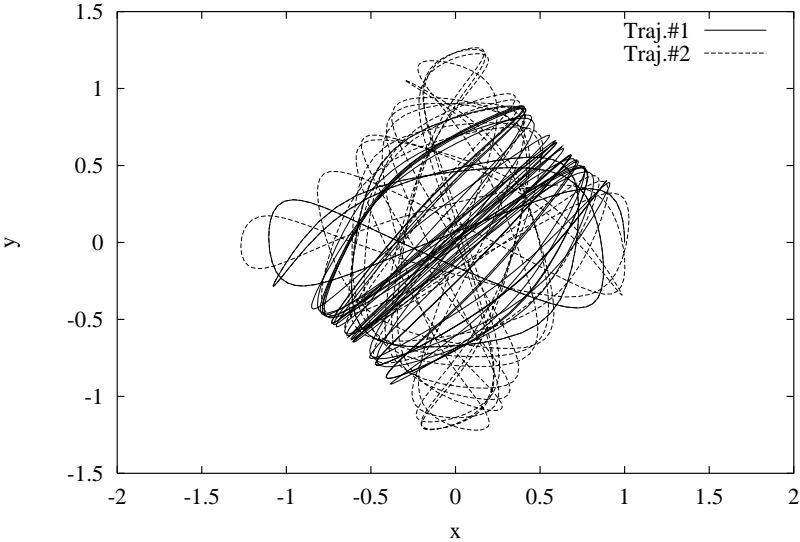
The divergence of the trajectories



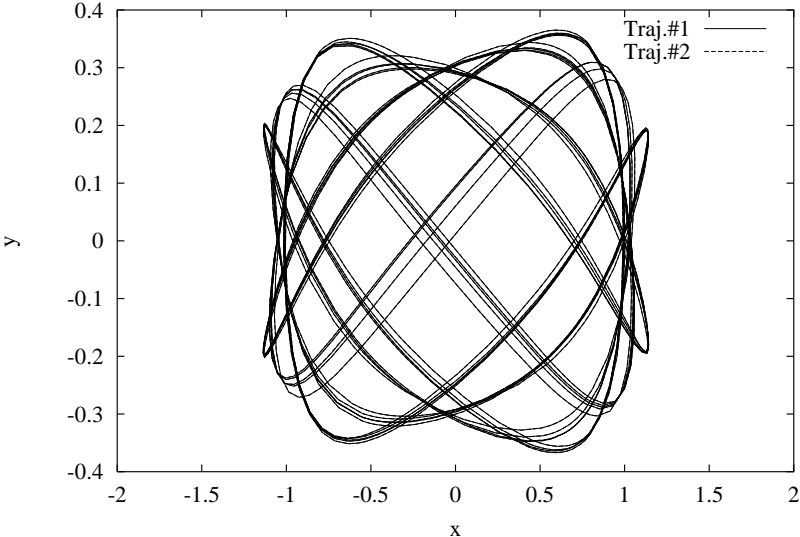
and the classical Poincaré section



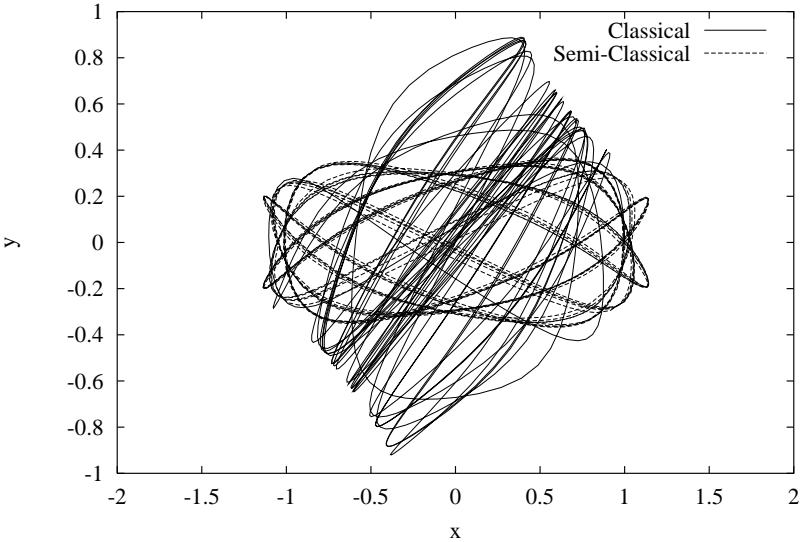
These effects are stronger for $k = 1$. Classical:



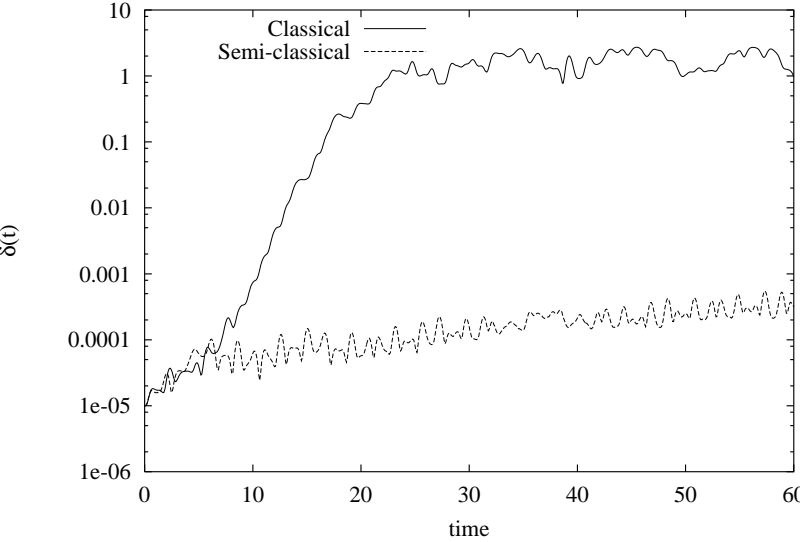
And “quantum” (semi-classical)



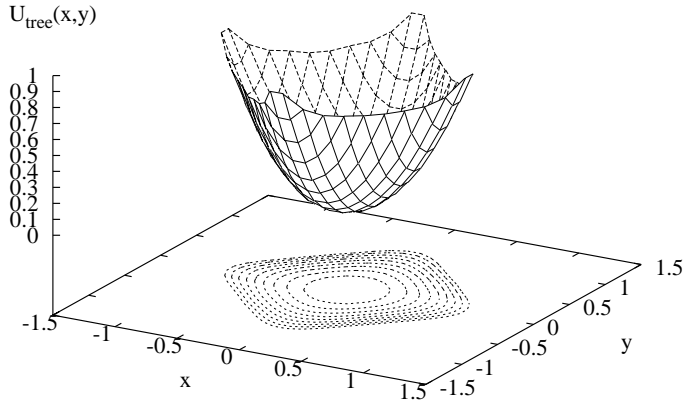
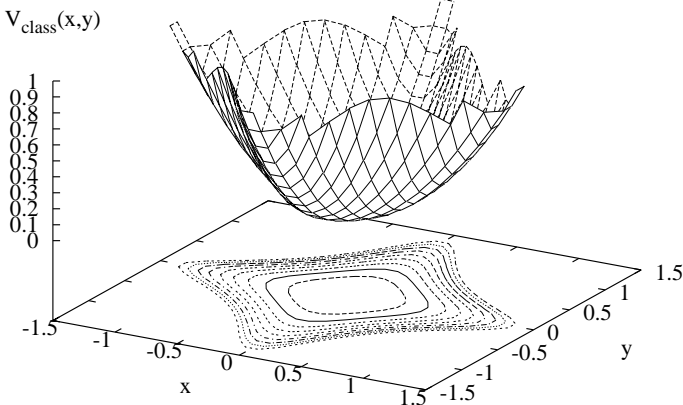
The two types of orbit evolve differently



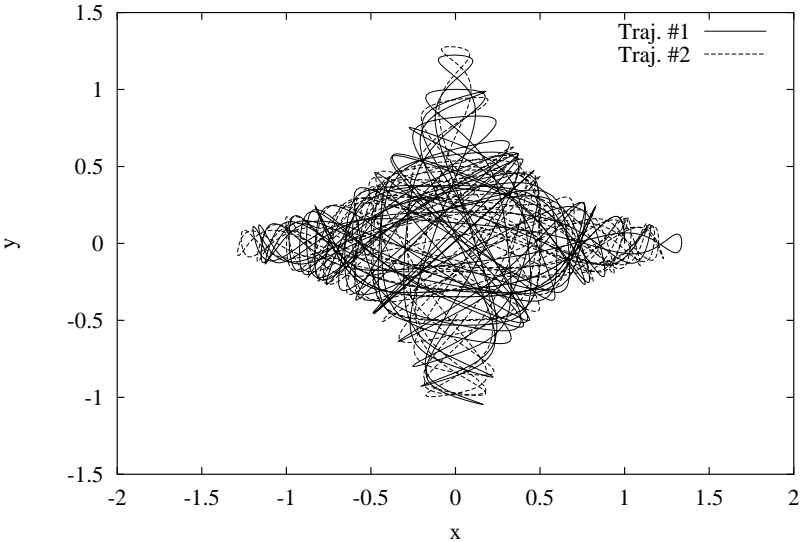
The phase space distance $\delta(t)$ shows it well



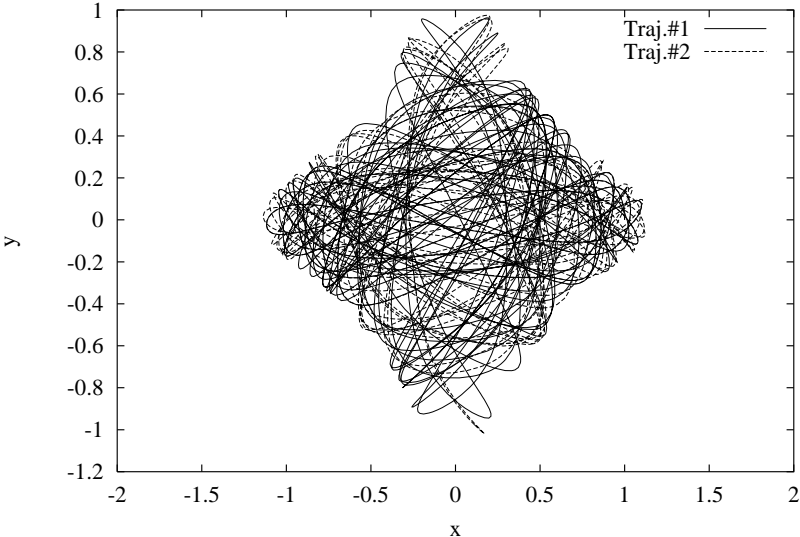
This behavior comes from the marked difference between $V_{class}(x, y)$ and $U_{tree}(x, y)$



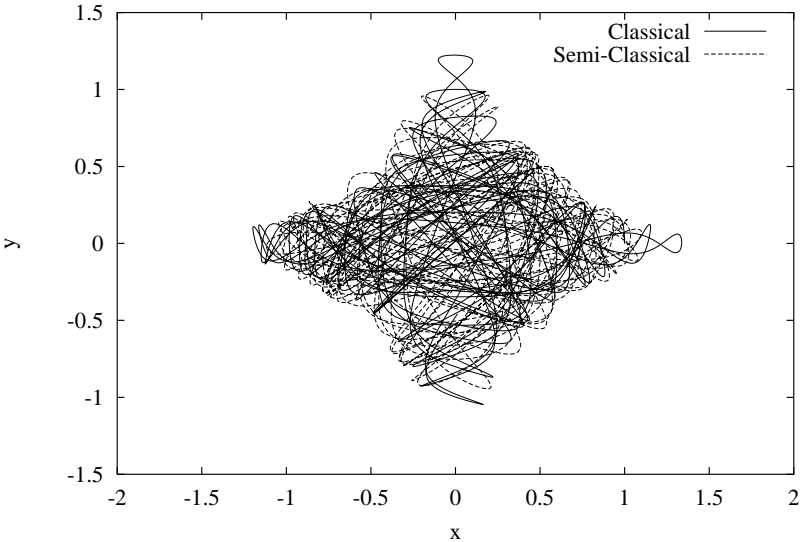
At $k = 4$ both systems are chaotic. Classical:



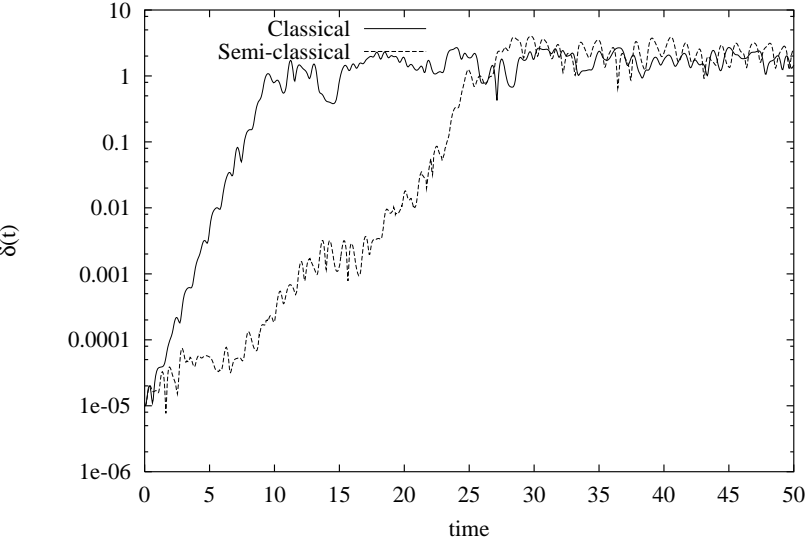
and quantum orbits:



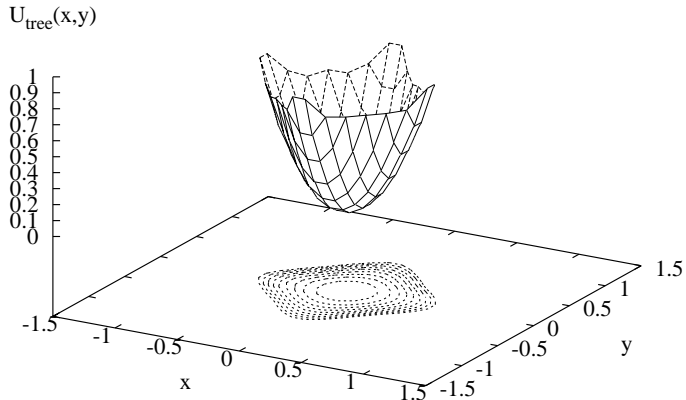
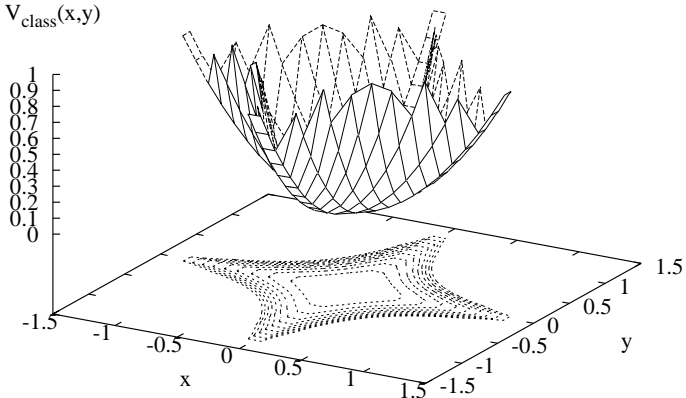
A side by side comparison is instructive



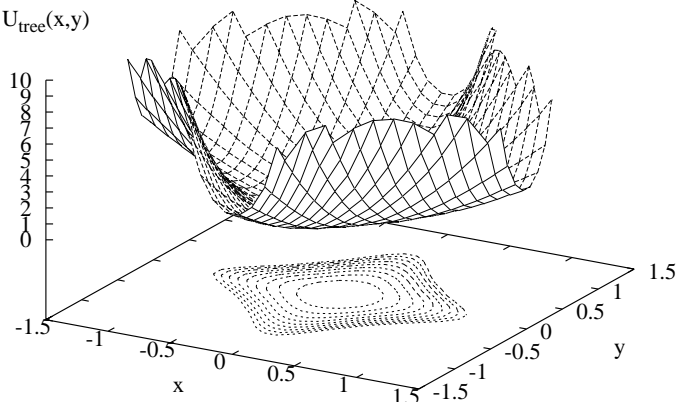
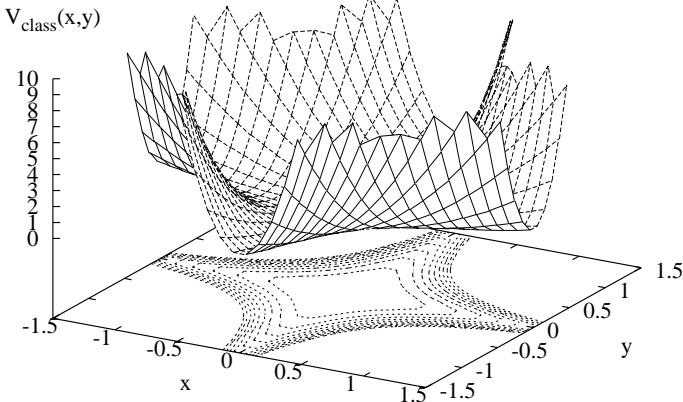
Still some quantum suppression of chaos



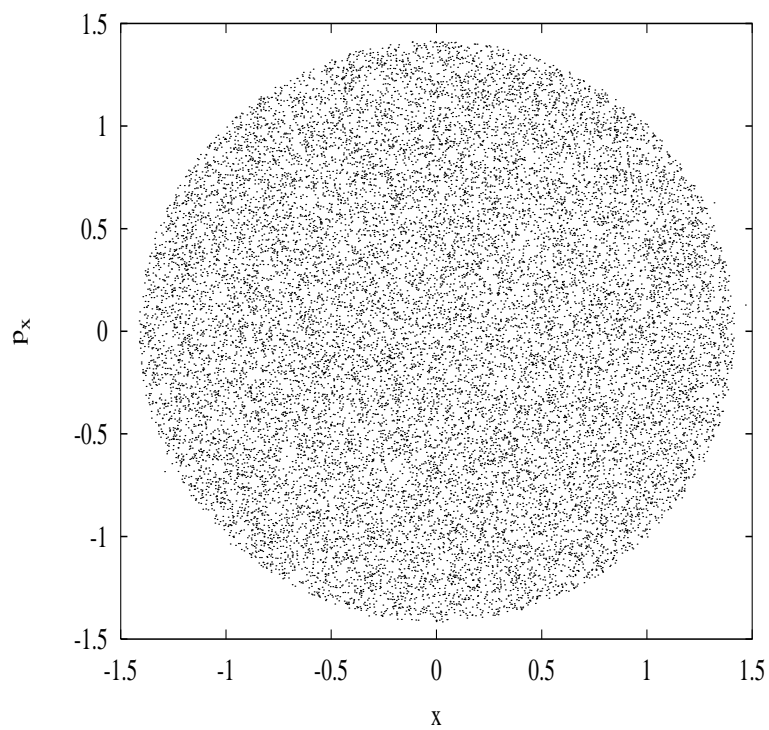
The difference in the potentials is quite clear



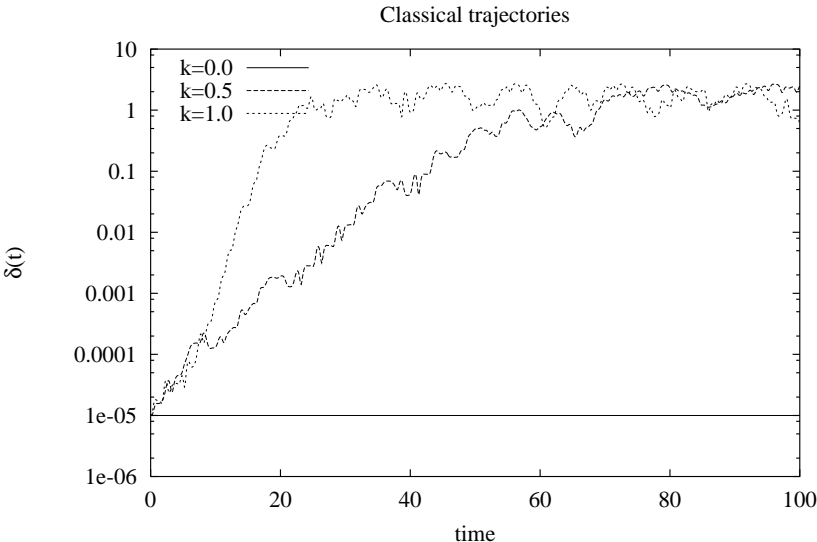
And with a different vertical scale



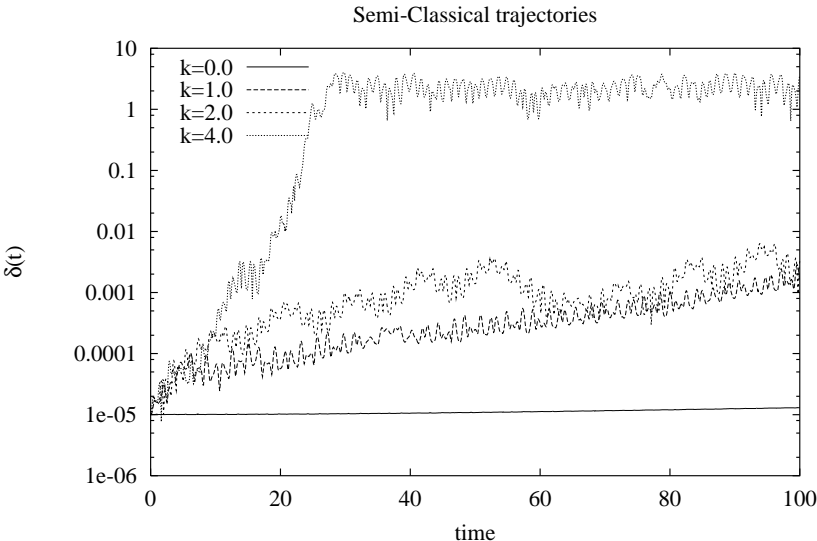
At $k = 4$ the classical system is fully chaotic, as the Poincaré surface of section shows well



A global look at orbit divergence. Classical



and semi-classical



Conclusions and outlook

- U_{tree} provides non-trivial quantum corrections to the classical behavior.
- A quantum suppression of chaos is observed, but not a complete elimination.
- A Poincaré section study of U_{tree} would be useful to clarify how phase space is globally affected by the quantum effects.
- A 2- d WKB? Interesting but immensely difficult in fully chaotic regimes: one needs *all* periodic orbits of the system (they are dense in phase space).
- However, *maybe* for small k the quantum suppression of chaos is enough to allow the WKB approach. Interesting...